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GENERAL NONCONVEX WIENER-HOPF EQUATIONS AND GENERAL NONCONVEX VARIATIONAL INEQUALITIES

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Abstract

This paper presents a Wiener-Hopf equations technique to suggest and analyze an iterative algorithm for solving the strongly nonlinear general nonconvex variational inequality. We establish the equivalence between the strongly nonlinear general nonconvex variational inequalities and the general nonconvex Wiener-Hopf equations. We also prove the convergence of the suggested algorithm under suitable conditions. Some special cases are also discussed.

1. Introduction

This work was inspired by the variational inequalities theory introduced by Stampacchia [16], this theory provides simple and unified framework to study a wide class of problems arising in pure and applied sciences. The existence and iterative schemes of variational inequalities

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have been investigated over convex sets, and that is due to the fact that all techniques are mainly based on the properties of the projection operator over convex sets. Recently, a new class of nonconvex sets, called uniformly prox-regular sets, has been introduced and studied in [5]. In [4], Bounkhel et al. introduced a new class of variational inequalities called the nonconvex variational inequalities. Noor [10], Moudafi [9], and Pang et al. [13] have also considered the variational inequality problems over these nonconvex sets. In [10-12], Noor has shown that the projection technique can be extended to nonconvex variational inequalities and has established the equivalence between the nonconvex variational inequalities and fixed point problems by using the projection technique. This equivalent alternative formulation has been used to investigate the existence of a solution of the nonconvex variational inequalities on one hand and to introduce some iterative methods on the other hand.

In this paper, a new class of nonconvex variational inequalities involving three nonlinear operators, is introduced and it is called the strongly nonlinear general nonconvex variational inequality. Also, a new Wiener-Hopf equations technique was applied to solve this new class of variational inequalities.

For more information about applications, numerical methods and other aspects of variational inequalities, one may refer to [1-16].

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle ., . \rangle$ and $\| . \|$, respectively. Let K be a nonempty closed subset in H. One recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [5, 6, 14].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : \exists \alpha > 0 \text{ s.t. } u \in P_K[u + \alpha\xi]\},$$

where

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is,

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. *Let K be a nonempty, closed subset in H. Then* $\xi \in N_K^P(u)$, *if and only if there exists a constant* $\alpha > 0$ *such that*

$$\langle \xi, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K.$$

Definition 2.2. ([6]). The Clarke normal cone, denoted by $N_K^C(u)$, is defined as

$$N_K^C(u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}[S]$ denotes the closure of the convex hull of S. One always has $N_K^P(u) \subset N_K^C(u)$. The converse is not true in general. Note that $N_K^C(u)$ is always a closed and convex cone and that $N_K^P(u)$ is always a convex cone, but may be nonclosed (see [5, 6]). Furthermore, if K is convex then all the existing normal cones and the normal cone in the sense of convex analysis $N_K(u)$ given by

$$N_K(u) := \{v \in H : \langle v, u^* - u \rangle, \text{ for all } v \in K\},$$

are coincided. A new class of nonconvex sets, called uniformly r-prox-regular sets has been introduced and studied in [5]. It has been successfully used in many nonconvex applications such as optimization, economic models, dynamical systems, and differential inclusions. This class seems particularly well suited to overcome the difficulties, which arise due to the nonconvexity assumption on K, see [4].

Definition 2.3. ([5]). For a given $r \in (0, \infty]$, a subset K is said to be uniformly r-prox-regular, if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N_K^P(u)$, one has

$$\langle \xi / \|\xi\|, v - u \rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

Recall that for $r = +\infty$, the uniform r-prox-regularity of K is equivalent to the convexity of K. The following lemma summarizes some important consequences of the uniform-prox-regularity needed in the sequel.

Lemma 2.2. *Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K is uniformly r-prox-regular, then the following holds*:

(i) $\forall u \in K_r$, $P_K(u) \neq \varnothing$;

(ii) $\forall r' \in (0, r)$, *the operator P_K is Lipschitz continuous with constant $\frac{r}{r - r'}$ on $K_{r'}$.*

For given nonlinear operators T, A, g, we consider the problem of finding $u \in H : g(u) \in K$ such that

$$\langle Tu,\ g(v) - g(u)\rangle + \lambda\|g(v) - g(u)\|^2 \geq \langle A(u),\ g(v) - g(u)\rangle,$$
$$\forall v \in H : g(v) \in K, \tag{1}$$

Inequality of type (1) is called the strongly nonlinear general nonconvex variational inequality SNGNVI, and λ is a positive parameter. If $g = I$, the identity operator, then (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle + \lambda\|v - u\|^2 \geq \langle A(u), v - u\rangle, \quad \forall v \in K, \tag{2}$$

which is known as the strongly nonlinear nonconvex variational inequality, and was introduced and studied by Noor [11]. If $A(u) \equiv 0$, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle + \lambda\|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in H : g(v) \in K, \tag{3}$$

which is called the general nonconvex variational inequality. If $g \equiv I$, the identity operator, and $\lambda = 0$, then problem (3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{4}$$

Problem (4) is the variational inequality introduced and studied by Stampacchia [16].

If K is a nonconvex (uniformly r-prox-regular) set, then problem (1) is equivalent to finding $u \in K$ such that

$$0 \in Tu - A(u) + N_K^P(g(u)), \tag{5}$$

where $N_K^P(g(u))$ denotes the normal cone of K at $g(u)$ in the sense of nonconvex analysis. Problem (5) is called the nonconvex variational inclusion problem associated with the nonconvex variational inequality (1). This implies that the variational inequality (1) is equivalent to finding a zero of the sum of two monotone operators (5).

3. Iterative Algorithm

In this section, we establish the equivalence between the nonconvex variational inequality SNGNVI (1) and the fixed point problem by using the projection operator technique used in Noor [10-12].

Lemma 3.1. *$u \in K$ is a solution of the strongly nonlinear nonconvex variational inequality* (1), *if and only if $u \in K$ satisfies the relation*

$$g(u) = P_K[g(u) - \rho Tu + \rho A(u)], \tag{6}$$

where P_K is the projection of H onto the uniformly r-prox-regular set K.

Proof. Let $u \in K$ be a solution of (1). Then, for a constant $\rho > 0$,

$$0 \in g(u) + \rho N_K^P(g(u)) - (g(u) - \rho(Tu - A(u)))$$

$$= (I + \rho N_K^P)g(u) - (u - \rho(Tu - A(u)))$$

$$\Leftrightarrow$$

$$g(u) = (I + \rho N_K^P)^{-1}[g(u) - \rho Tu + \rho A(u)]$$

$$= P_K[g(u) - \rho Tu + \rho A(u)],$$

where we have used the well-known fact that $P_K \equiv (I + N_K^P)^{-1}$. □

Lemma 3.1 implies that the strongly nonlinear general nonconvex variational inequality (1) is equivalent to the fixed point problem (6). This alternative equivalent formulation is very useful from the numerical and theoretical point of views. The fixed point problem (6) is used to suggest the following iterative method for solving the SNGNVI (1).

We now consider the problem of solving the nonconvex Wiener-Hopf equations. To be more precise, let P_K be the projection of H onto the nonconvex set K and $Q_K = I - P_K$, where I is the identity operator. For given nonlinear operators $T, A, g,$ consider the problem of finding $z \in H$ such that

$$Tg^{-1}P_K z + \rho^{-1}Q_K z = A(g^{-1}P_K z), \tag{7}$$

where we have used the fact that g^{-1} exists. Equation (7) is called the strongly nonlinear nonconvex Wiener-Hopf equation. For some special value of the operators $T, A, g,$ one can obtain the original Wiener-Hopf equations, considered by Shi [15].

Now, we use Lemma 3.1 to establish the equivalence between problems (1) and (7) and this is the main motivation of our next result.

Lemma 3.2. *The nonconvex Wiener-Hopf equation* (7) *has a solution* $z \in H$, *if and only if the strongly nonlinear nonconvex variational inequality* (1) *has solution* $u \in K$, *provided*

$$u = g^{-1}P_K z,$$

$$z = g(u) - \rho(Tu - A(u)), \tag{8}$$

where P_K *is the projection of H onto the closed nonconvex set K.*

Proof. Let $u \in K$ be a solution of (1). Then, from Lemma 3.1, one obtains

$$u = g^{-1}P_K[g(u) - \rho(Tu - A(u))].$$

Let

$$z = g(u) - \rho(Tu - A(u)),$$

then

$$u = g^{-1}P_K z.$$

Then, from (8), one has

$$z = P_K z - \rho T g^{-1} P_K z + \rho A(g^{-1}P_K z),$$

that is,

$$\rho^{-1}Q_K z + Tg^{-1}P_K z = A(g^{-1}P_K z).$$

This shows that $z \in H$ is a solution of (7) and the converse is also true. □

Algorithm 3.1. For a given $z_0 \in K$, find the approximate solution z_{n+1} by the iterative scheme

$$g(u_n) = P_K z_n, \quad n = 0, 1, 2, \ldots, \tag{9}$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n[g(u_n) - \rho Tu_n + \rho A(u_n)], \quad n = 0, 1, 2, \ldots, \tag{10}$$

where $0 \le \alpha_n \le 1$ for all $n \ge 0$.

4. Convergence

In this section, we will prove the convergence of the Algorithm 3.1, for this purpose, we need the following:

Definition 4.1. An operator $T : H \to H$ is said to be

(i) strongly monotone, if and only if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H;$$

(ii) Lipschitz continuous, if and only if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

Theorem 4.1. *Let* P_K *be the Lipschitz continuous operator with constant* $\delta = \frac{r}{r - r'}$. *Let* T, g *be strongly monotone with constant* $\alpha > 0$, $\eta > 0$, *respectively, and Lipschitz continuous with constant* $\beta > 0, \sigma > 0$, *respectively. Let the operator* A *be Lipschitz continuous with constant* $\gamma > 0$. *If there exists a constant* ρ *such that*

$$\left|\rho - \frac{(\alpha\delta - \gamma(1 - (1 + \delta)k))}{\delta(\beta^2 - \gamma^2)}\right| < \frac{\sqrt{(\alpha\delta - \gamma(1 - (1 + \delta)k))^2 - (\beta^2 - \gamma^2)(\delta^2 - (1 - (1 + \delta)k)^2)}}{\delta(\beta^2 - \gamma^2)}, \quad (11)$$

$$\delta\rho\alpha > 1,\ k < 1,\ k = \sqrt{1 - 2\eta + \sigma^2},\ \delta\alpha > \gamma(1 - (1 + \delta)k) + \sqrt{(\beta^2 - \gamma^2)(\delta^2 - (1 - (1 + \delta)k)^2)},$$

and $\alpha_n \in [0, 1], \forall n \geq 0; \sum_{n=0}^{\infty} \alpha_n = \infty$, *then the approximate solution* z_n *obtained from Algorithm* 3.1 *converges to a solution* $z \in H$ *satisfying the nonconvex Wiener-Hopf equation* (7).

Proof. Let $z \in H$ be a solution of (7). Then, using Lemma 3.2, we get

$$z = (1 - \alpha_n)z + \alpha_n\{g(u) - \rho(Tu - A(u))\}, \tag{12}$$

where $0 \le \alpha_n \le 1$.

From (10) and (12), one has

$$\|z_{n+1} - z\| \le (1 - \alpha_n)\|z_n - z\| + \alpha_n\|u_n - u - (g(u_n) - g(u))\|$$
$$+ \alpha_n\|u_n - u - \rho(Tu_n - Tu)\| + \alpha_n\rho\|A(u_n) - A(u)\|. \tag{13}$$

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\|u_n - u - \rho(Tu_n - Tu)\|^2 \le \|u_n - u\|^2 - 2\rho\langle u_n - u, Tu_n - Tu\rangle + \rho^2\|Tu_n - Tu\|^2$$
$$\le (1 - 2\rho\alpha + \rho^2\beta^2)\|u_n - u\|^2. \tag{14}$$

Similarly, since the operator g is strongly monotone with constant $\eta > 0$ and Lipschitz continuous with constant $\sigma > 0$, it follows that

$$\|u_n - u - (g(u_n) - g(u))\|^2 \le \|u_n - u\|^2 - 2\langle u_n - u, g(u_n) - g(u)\rangle + \|g(u_n) - g(u)\|^2$$
$$\le (1 - 2\eta + \sigma^2)\|u_n - u\|^2. \tag{15}$$

From (13), (14), (15), and using the Lipschitz continuity of the operator A with constant $\gamma > 0$, we get

$$\|z_{n+1} - z\| \le (1 - \alpha_n)\|z_n - z\|$$
$$+ \alpha_n\left(\sqrt{1 - 2\eta + \sigma^2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\gamma\right)\|u_n - u\|, \tag{16}$$

from (9) and the Lipschitz continuity of the projection operator P_K with constant δ, we have

$$\|u_n - u\| = \|u_n - u - (g(u_n) - g(u)) + (P_K z_n - P_K z)\|$$
$$\le \sqrt{1 - 2\eta + \sigma^2}\|u_n - u\| + \delta\|z_n - z\|,$$

$$\|u_n - u\| \leq \frac{\delta}{(1-k)} \|z_n - z\|,$$

where $k = \sqrt{1 - 2\eta + \sigma^2}$, and we can rewrite (16) as

$$\begin{aligned} \|z_{n+1} - z\| &\leq (1 - \alpha_n) \|z_n - z\| \\ &\quad + \frac{\alpha_n \delta}{(1-k)} \left(k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\gamma \right) \|z_n - z\| \\ &\leq (1 - \alpha_n) \|z_n - z\| + \alpha_n \theta \|z_n - z\|, \end{aligned}$$

where

$$\theta = \frac{\delta}{(1-k)} \left(k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\gamma \right). \tag{17}$$

From (11), we see that $\theta < 1$ and consequently,

$$\begin{aligned} \|z_{n+1} - z\| &\leq [(1 - (1 - \theta))\alpha_n] \|z_n - z\| \\ &\leq \prod_{i=0}^{n} [(1 - (1 - \theta))\alpha_i] \|z_0 - z\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, it follows that $\lim_{n \to \infty} \prod_{i=0}^{n} [(1 - (1 - \theta))\alpha_i] = 0$. Consequently, the sequence $\{z_n\}$ converges strongly to z in H satisfying the nonconvex Wiener-Hopf equation (7). $\square$

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